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Some Series of Rectangular Designs

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Summary

The present paper describes some methods of constructing rectangular designs.
Key words : L2 – designs; rectangular association scheme.

Introduction

A rectangular design is a three associate partially balanced incomplete block design based on the rectangular association scheme introduced by Vartak [5]. For the definition of these designs see e.g. Dey [1] (p. 203).

Several methods for constructing rectangular designs are available in literature, see for example, Ghosh and Das [2], Goswami, Majumdar and Pal [3], Kageyama and Mohan [4], etc. In the present paper, some more methods of constructing rectangular designs are presented.

2. *Methods of constructing rectangular designs*

Theorem 2.1: There always exists rectangular designs with parameters

$$v = mn,\ b = \binom{m}{s}\binom{n}{t},\ r = \binom{m-1}{s-1}\binom{n-1}{t-1},\ k = st,\ \lambda_1 = \binom{m-1}{s-1}\binom{n-2}{t-2}$$

$$\lambda_2 = \binom{m-2}{s-2}\binom{n-1}{t-1} \quad \text{and} \quad \lambda_3 = \binom{m-2}{s-2}\binom{n-2}{t-2} \qquad (2.1)$$

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$$\lambda_2 = \binom{m-1}{s-1}\binom{n-2}{t-2} \quad \text{and} \quad \lambda_3 = \binom{m-2}{s-2}\binom{n-2}{t-2} \tag{2.2}$$

for any integers m, s, n, t, $m \geq s \geq 2$ and $n \geq t \geq 2$.

Proof : Let v = mn be the number of symbols arranged in a rectangular matrix A with m rows and n columns. By selecting all possible submatrices of order $s \times t$ and writing the elements of each sub matrix as a block of size st, one can get the rectangular design with parameters given in (2.1).

Similarly by arranging the v symbols in a rectangular matrix A^* with n rows and m columns and taking all possible $t \times s$ submatrices one can obtain the rectangular design with parameters given in (2.2).

Corollary 2.1 : If s = m–1 and t = n–1 in (2.1) and (2.2), then the resulting rectangular designs are symmetric with parameters

$$v = mn = b,\ r = (m-1)(n-1) = k,\ \lambda_1 = (m-1)(n-2)$$

$$\lambda_2 = (m-2)(n-1) \text{ and } \lambda_3 = (m-2)(n-2) \tag{2.3}$$

$$v = mn = b,\ r = (m-1)(n-1) = k,\ \lambda_1 = (m-2)(n-1)$$

$$\lambda_2 = (m-1)(n-2) \text{ and } \lambda_3 = (m-2)(n-2) \tag{2.4}$$

Corollary 2.2 : If m = n and s = t in (2.1) and (2.2), then the resulting designs are two associate class L2-designs with parameters

$$v = m^2,\ b = \binom{m}{s}^2,\ r = \binom{m-1}{s-1}^2,\ k = s^2,\ \lambda_1 = \binom{m-1}{s-1}\binom{m-2}{s-2}$$

and $\lambda_2 = \binom{m-2}{s-2}^2$ provided $m > s \geq 2$. (2.5)

Corollary 2.3 : If s = m–1 in (2.5), then the resulting L_2 – design is a symmetric with parameters

$$v = m^2 = b,\ r = (m-1)^2 = k,\ \lambda_1 = (m-1)(m-2) \text{ and}$$
$$\lambda_2 = (m-2)^2 \tag{2.6}$$

Theorem 2.2 : There always exists rectangular designs with parameters

$$v = mn,\ b = mn(n-1),\ r = m(n-1),\ k = m,\ \lambda_1 = 0,\ \lambda_2 = n-1$$

$$\text{and } \lambda_3 = 2 \qquad (2.7)$$

$$v = mn,\ b = mn(n-1),\ r = m(n-1),\ k = m,\ \lambda_1 = n-1,\ \lambda_2 = 0$$

$$\text{and } \lambda_3 = 2 \qquad (2.8)$$

Proof : Let v = mn be the number of symbols arranged in a rectangular matrix with m rows and n columns. Let a_{ij} be the element in the (i, j)th position of A.

For each entry a_{ij}, omit the jth column; replace the remaining (n–1) elements in the ith row by a_{ij} and form each column as a block of size m. This way one can have mn(n–1) blocks of size m, which is the rectangular design with the parameters given in (2.7).

Similarly arrange the v symbols in a rectangular matrix A^* with n rows and m columns. For each a^*_{ij} omit the ith row replace the remaining (n–1) elements in the jth column and form each row as a block one may get the rectangular design with parameters given in (2.8).

Theorem 2.3: There always exists rectangular designs with parameters

$$v = mn,\ b = mn^2,\ r = mn,\ k = m,\ \lambda_1 = 0,\ \lambda_2 = m+n-1 \text{ and}$$
$$\lambda_3 = 2 \qquad (2.9)$$

$$v = mn,\ b = mn^2,\ r = mn,\ k = m,\ \lambda_1 = m+n-1,\ \lambda_2 = 0 \text{ and}$$
$$\lambda_3 = 2 \qquad (2.10)$$

Proof: Let A and A^* be the two rectangular matices of order m× n and n× m respectively as defined in Theorem 2.2.

For each of a_{ij}, replace the ith row by a_{ij} and form each column as a block. This leads to mn^2 blocks of size m, which constitute the rectangular design with parameters given in (2.9).

In a similar manner, one may construct the rectangular design with parameters given in (2.10) from the matix A^*.

Theorem 2.4: There always exists rectangular designs with parameters

$$v = mn,\ b = n(mn-m+1),\ r = mn-m+1,\ k = m,\ \lambda_1 = 0,\ \lambda_2 = n \text{ and } \lambda_3 = 2 \quad (2.11)$$

$$v = mn,\ b = n(mn-m+1),\ r = mn-m+1,\ k = m,\ \lambda_1 = n,\ \lambda_2 = 0 \text{ and } \lambda_3 = 2 \quad (2.12)$$

Proof: The rectangular design with parameters (2.11) is obtained by taking each column of the matrix A as a block together with the rectangular design with parameters (2.7).

Similarly the rectangular design with parameters (2.12) is obtained by taking each row of the matrix A^* as a block together with the rectangular design with parameters (2.8).

Example. Let m = 3, n = 4, s = 2 and t = 3. Then the parameters of the rectangular designs given in (2.3) and (2.4) denoted by D_1 and D_2 are as follows:

$$v = 12 = b,\ r = 6 = k,\ \lambda_1 = 4,\ \lambda_2 = 3,\quad \lambda_3 = 2$$

$$v = 12 = b,\ r = 6 = k,\ \lambda_1 = 3,\ \lambda_2 = 4;\quad \lambda_3 = 2$$

The matrics A and A^* and the rectangular designs D_1 and D_2 are obtained as

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}, \qquad A^* = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 1 & 2 & 3 & 5 & 6 & 7 \\ 1 & 2 & 3 & 9 & 10 & 11 \\ 5 & 6 & 7 & 9 & 10 & 11 \\ 1 & 2 & 4 & 5 & 6 & 8 \\ 1 & 2 & 4 & 9 & 10 & 12 \\ 5 & 6 & 8 & 9 & 10 & 12 \\ 1 & 3 & 4 & 5 & 7 & 8 \\ 1 & 3 & 4 & 9 & 11 & 12 \\ 2 & 3 & 4 & 6 & 7 & 8 \\ 2 & 3 & 4 & 10 & 11 & 12 \\ 6 & 7 & 8 & 10 & 11 & 12 \\ 5 & 7 & 8 & 9 & 11 & 12 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 2 & 4 & 5 & 7 & 8 \\ 1 & 2 & 4 & 5 & 10 & 11 \\ 1 & 2 & 7 & 8 & 10 & 11 \\ 4 & 5 & 7 & 8 & 10 & 11 \\ 1 & 3 & 4 & 6 & 7 & 9 \\ 1 & 3 & 4 & 6 & 10 & 12 \\ 1 & 3 & 7 & 9 & 10 & 12 \\ 4 & 6 & 7 & 9 & 10 & 12 \\ 2 & 3 & 5 & 6 & 8 & 9 \\ 2 & 3 & 8 & 9 & 11 & 12 \\ 5 & 6 & 8 & 9 & 11 & 12 \\ 2 & 3 & 5 & 6 & 11 & 12 \end{bmatrix}$$

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